Lecture 6. Numerical Integration

6.1 Trapezoid Rule

A similar approach is much better. We approximate the area under a curve over a small interval as the area of a trapezoid. This technique for approximating an integral is known as the ***Trapezoid Rule***. In we see an area under a curve approximated by rectangles and by trapezoids; it is apparent that the trapezoids give a substantially better approximation on each subinterval.

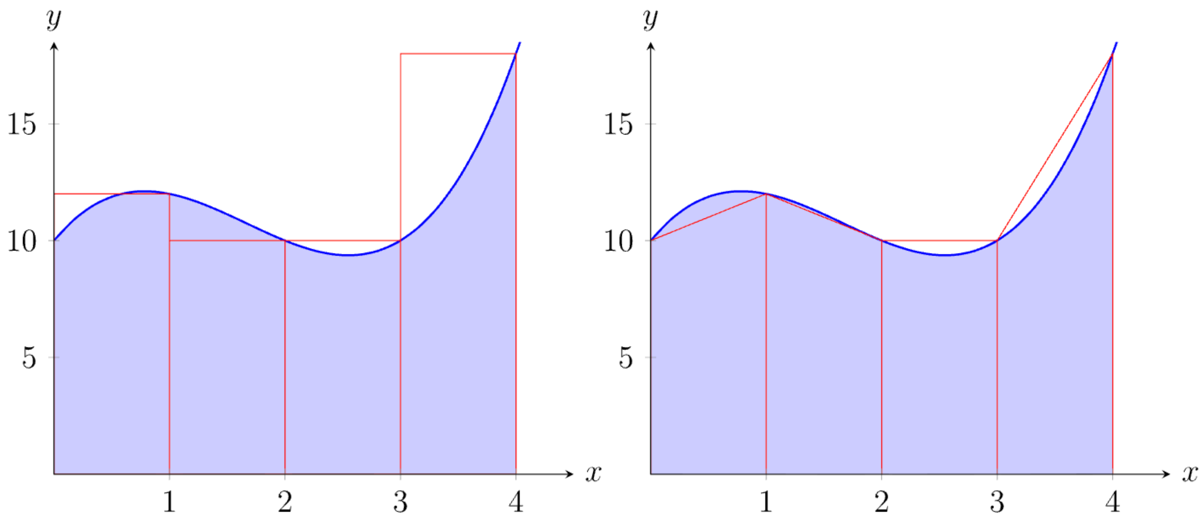

Figure 6.1 Approximating an area with rectangles and with trapezoids.

As with rectangles, we divide the interval into n equal subintervals of length Δx. A typical trapezoid is pictured in Figure 6.1 it has area

$$\frac{f(x0)+f(x1)}{2}\Delta x+\frac{f(x1)+f(x2)}{2}\Delta x+\cdots+\frac{f(xn-1)+f(xn)}{2}\Delta x$$

$$=\frac{f(x0)}{2}+f(x1)+f(x2)+\cdots+f(xn-1)+\frac{f(xn)}{2}\Delta x$$

$$=\frac{\Delta x}{2}(f(x0)+2f(x1)+2f(x2)+\cdots+2f(xn-1)+f(xn)).$$

For a modest number of subintervals this is not too difficult to do with a calculator. a computer can easily handle many subintervals.

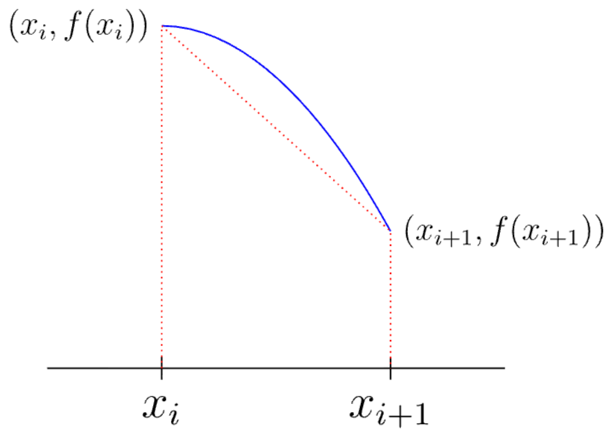

Figure 2: A single trapezoid.

We summarize this result in the theorem below.

Theorem 2.45. Trapezoid Rule. *Let $f(x)$ be defined on a closed interval $[a,b]$ that is subdivided into n subintervals of equal length $\Delta x = (b-a)/n$ using $n+1$ points $x_i = a + i\Delta x$:*

$$x_0 = a, x_1 = a + \Delta x, \ldots, x_{n-1} = a + (n-1)\Delta x, x_n = b.$$

Then the integral $\int_a^b f(x)\,dx$ can be approximated by

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right).$$

6.2 Simpson's Rule

The trapezoid approximation works well, especially compared to rectangles. because the tops of the trapezoids form a reasonably good approximation to the curve when Δx or h is fairly small. What if we try to approximate the curve more closely by using something other than a straight line in our search for a better approximation to the integral of f The obvious candidate is a parabola as shown in Figure 6.3. If we can approximate a short piece of the curve with a parabola with equation $y = ax^2 + bx + c$. we can easily compute the area under the parabola.

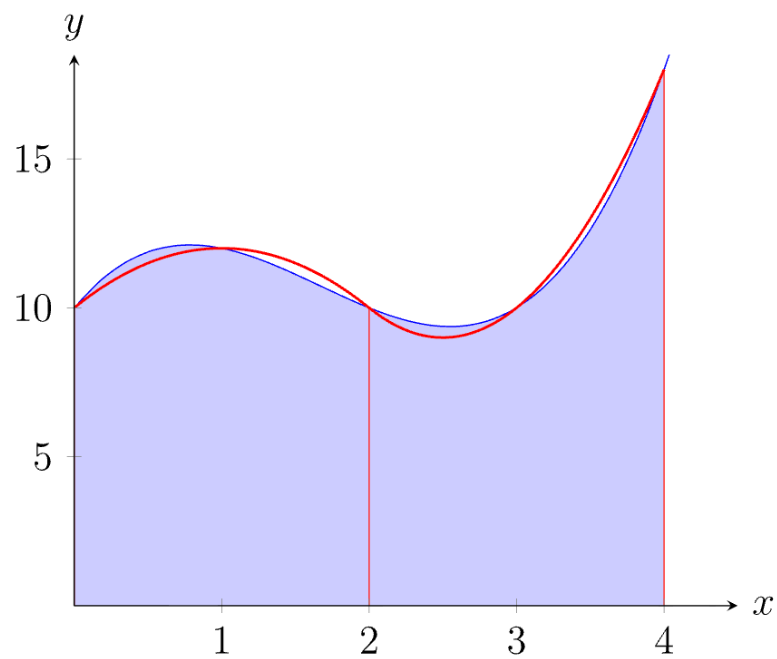

Figure 6.3. Approximating an area with parabolas.

Theorem 2.48. Simpson's Rule. *Let $f(x)$ be defined on a closed interval $[a,b]$ that is subdivided into n even subintervals of equal length $\Delta x = (b-a)/n$ using $n+1$ points $x_i = a + i\Delta x$:*

$$x_0 = a, x_1 = a + \Delta x, \ldots, x_{n-1} = a + (n-1)\Delta x, x_n = b.$$

Then the integral $\int_a^b f(x)\,dx$ can be approximated by

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 4f(x_{n-1}) + f(x_n)\right).$$

Example. 1

Find the value of the integral $\int_0^1 \frac{1}{1+x}\,dx$ with h=0.1 using trapezoidal rule.

Solution:

For step size $h = 0.1$ determine the number of intervals n and the x-values

$n = \frac{b-a}{h} = \frac{1-0}{0.1} = 10$ and $f(x) = \frac{1}{1+x}$

at each of these points.

x	$f(x) = \frac{1}{1+x}$	$2f(x)$
$x_0 = 0.0$	$1/(1+0.0) = 1$	
$x_1 = 0.1$		2*0.909091
$x_2 = 0.2$		2*0.833333
$x_3 = 0.3$		2*0.769231
$x_4 = 0.4$		2*0.714286
$x_5 = 0.5$		2*0.666667
$x_6 = 0.6$		2*0.625000
$x_7 = 0.7$		2*0.588235
$x_8 = 0.8$		2*0.555556
$x_9 = 0.9$		2*0.526316
$x_{10} = 1.0$	$1/(1+1.0) = 0.5$	
$\sum$	$\mathbf{1.5}$	$\mathbf{12.37543}$

The formula is:

$$\int_a^b f(x)dx \approx \frac{h}{2}[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

Substitute the values:

$$I \approx \frac{0.1}{2}[f(0) + 2(f(0.1) + f(0.2) + \cdots + f(0.9)) + f(1.0)]$$

$$I \approx 0.05[1.5 + 12.37543] \approx 0.6937715$$

The value of the integral using the trapezoidal rule with $h = 0.1$ is approximately 0.6938.

Example. 2

Evaluate the integral $\int_0^1 \frac{\sin x}{x} dx$ with h=0.1 using trapezoidal rule.

Solution:

1. The number of intervals is $n = \frac{1-0}{0.2} = 5$

$$I \approx \frac{h}{2}[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)]$$

We calculate the function values at each point (using radians):

x	$f(x) = \frac{\sin x}{x}$	$2f(x)$
$x_0 = 0.0$	$\lim_{x\to 0} \frac{\sin x}{x} = 1$	
$x_2 = 0.2$		2*0.993345
$x_4 = 0.4$		2*0.973545
$x_6 = 0.6$		2*0.941071
$x_8 = 0.8$		2*0.896695
$x_{10} = 1.0$	$\frac{\sin(1.0)}{1.0} \approx 0.841471$	
$\sum$	1.841471	7.609312

Now, substitute these values into the formula:

$$I_{h=0.2} \approx \frac{0.2}{2}[1.841471 + 7.609312] = 0.945078$$

2. Evaluate the integral using the trapezoidal rule with $h = 0.1$ The number of intervals is $n = \frac{1-0}{0.1} = 10$

$$I_{h=0.1} \approx \frac{0.1}{2}[f(0) + 2\sum_{i=1}^{9} f(x_i) + f(1)]$$

$$I_{h=0.1} \approx 0.05[1 + 2(0.998334 + 0.993345 + 0.985067 + 0.973545 + 0.958851 + 0.941071 + 0.920311 + 0.896695 + 0.870363) + 0.841471]$$

$$I_{h=0.1} \approx 0.05[1.841471 + 2(8.537582)]0.945832$$

Then:
The value of the integral is approximately **0.9451** for $h = 0.2$ and **0.9458** for $h = 0.1$

Example. 3

Find the value of the integral $\int_0^1 \frac{1}{1+x} dx$ with h=0.1 using **Simpson's** rule.

The number of intervals, n, is given by

$$n = \frac{b-a}{h}\frac{1-0}{0.1} = 10$$

is an even number, Simpson's 1/3 rule can be applied. The x-values from
We evaluate the function $f(x) = \frac{1}{1+x^2}$ at each of these points.

x	$f(x) = \dfrac{1}{1+x^2}$	$4\,f(x)$	$2\,f(x)$
0.0	1.000000		
0.1		4*0.990099	
0.2			2*0.961538
0.3		4*0.917431	
0.4			2*0.862069
0.5		4*0.800000	
0.6			2*0.735294
0.7		4*0.671141	
0.8			2*0.609756
0.9		4*0.552486	
1.0	0.500000		
$\sum$	1.5	15.724628	6.337314

Apply Simpson's 1/3 Rule formula

$$I \approx \frac{h}{3}[f(x_0) + 4(f(x_1) + f(x_3) + \cdots) + 2(f(x_2) + f(x_4) + \cdots) + f(x_n)]$$

Substitute the values:

$$I \approx \frac{0.1}{3}[1.5 + 15.724628 + 6.337314] = 0.785398$$

The value of the integral using Simpson's 1/3 rule with $h = 0.1$ is approximately 0.7854.